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Smooth shifts along trajectories of flows

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Abstract

Let Φ be a flow on a smooth, compact, finite-dimensional manifold M. Consider the subset $\mathcal{D}(\Phi)$ of $C^\infty(M,M)$ consisting of diffeomorphisms of M preserving the foliation of the flow Φ. Let also $\mathcal{D}_0(\Phi)$ be the identity path component of $\mathcal{D}(\Phi)$ with compact-open topology. We prove that under mild conditions on fixed points of Φ the space $\mathcal{D}_0(\Phi)$ is either contractible or homotopically equivalent to S^1.

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1. Introduction

Throughout, M will be a smooth (C^∞) connected manifold and $\mathcal{D}(M)$ be the space of diffeomorphisms of M. Let Φ be a smooth flow on M, and $\operatorname{Fix}\Phi$ be the fixed-point set of Φ. Define the map $\varphi : C^\infty(M,\mathbb{R}) \to C^\infty(M,M)$ by

$$\varphi(\alpha)(x) = \Phi\bigl(x, \alpha(x)\bigr),$$

where $\alpha \in C^\infty(M,\mathbb{R})$ and $x \in M$. We will say that φ is the *shift-map* along trajectories of Φ. If $\alpha \in C^\infty(M,\mathbb{R})$ and $f = \varphi(\alpha)$, then the following statements can easily be checked:

(1) f is homotopic to id_M;
(2) $f(\omega) \subset \omega$ for each trajectory ω of Φ. In particular, if $z \in \operatorname{Fix}\Phi$, then $f(z) = z$. Moreover,

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(3) f is a local diffeomorphism at z, i.e., the corresponding tangent map $f'(z): TM_z \to TM_z$ is an isomorphism.

Let $\mathcal{E}(\Phi) \subset C^\infty(M, M)$ be the set of all maps $f: M \to M$ satisfying the above conditions (2) and (3), $\mathcal{D}(\Phi)$ be the intersection $\mathcal{E}(\Phi) \cap \mathcal{D}(M)$, and $\mathcal{E}_0(\Phi)$ and $\mathcal{D}_0(\Phi)$ be the identity path components of the spaces $\mathcal{E}(\Phi)$ and $\mathcal{D}(\Phi)$ (respectively) in the compact-open topology. Evidently $\operatorname{Im}\varphi \subset \mathcal{E}_0(\Phi)$.

We use the map φ to study the homotopy types of the spaces $\mathcal{E}_0(\Phi)$ and $\mathcal{D}_0(\Phi)$. Take any $r = 0, 1, \ldots, \infty$ and endow $C^\infty(M, \mathbb{R})$ and $C^\infty(M, M)$ with the strong Whitney C^r-topologies and the spaces $\operatorname{Im}\varphi$, $\mathcal{D}_0(\Phi)$, and $\mathcal{E}_0(\Phi) \subset C^\infty(M, M)$ with the induced ones. The following theorem summarizes the results obtained in the paper.

Theorem 1. *Suppose for each fixed point z of Φ there exist local coordinates $(x_1, \ldots, x_m)$ and a linear flow Ψ on $\mathbb{R}^n$ $(n \leqslant m)$ such that $z = 0 \in \mathbb{R}^m$ and for all t in some neighborhood of $0 \in \mathbb{R}$, we have $p \circ \Phi_t = \Psi_t \circ p$, where $p: \mathbb{R}^m \to \mathbb{R}^n$ is the natural projection. Then*

(A) $\operatorname{Im}\varphi = \mathcal{E}_0(\Phi)$ *and* $\varphi: C^\infty(M, \mathbb{R}) \to \mathcal{E}_0(\Phi)$ *is either a homeomorphism or a covering map with $\mathbb{Z}$ as a group of covering slices. Also, the set $\varphi^{-1}(\mathcal{D}_0(\Phi))$ is* convex *if regarded as a subset of the* linear *space $C^\infty(M, \mathbb{R})$.*

(B) *If M is compact, then the inclusion $\mathcal{D}_0(\Phi) \subset \mathcal{E}_0(\Phi)$ is a homotopy equivalence and these spaces are either contractible or have the homotopy type of S^1. They are contractible whenever Φ has at least one non-closed trajectory or if the tangent linear flow at some fixed point of Φ is trivial.*

There are some applications of this theorem to Morse functions and Morse–Smale flows similar to [1]. We will consider them in the next papers.

The paper is organized as follows. In Section 2 we recall the definitions of Whitney topologies and some formulas concerning linear flows.

Section 3. We show that the set $Z_{\mathrm{id}} = \varphi^{-1}(\mathrm{id}_M)$ is a *subgroup* of $C^\infty(M, \mathbb{R})$ and for each $\alpha \in C^\infty(M, \mathbb{R})$, we have $\varphi^{-1}\varphi(\alpha) = \{\alpha + Z_{\mathrm{id}}\}$. Thus the correspondence $\{\alpha + Z_{\mathrm{id}}\} \mapsto \varphi(\alpha)$, where $\alpha \in C^\infty(M, \mathbb{R})$, is a bijection of the factor-group $C^\infty(M, \mathbb{R})/Z_{\mathrm{id}}$ onto the image $\operatorname{Im}\varphi$. Notice, however, that φ is not a homomorphism (see formulas (8) and (9)). We also describe Z_{id} in terms of the interior of $\operatorname{Fix}\Phi$ (Theorem 12). In particular, we obtain that $\operatorname{Fix}\Phi$ is nowhere dense in M, i.e., $\operatorname{Int}\operatorname{Fix}\Phi = \emptyset$ if and only if Z_{id} is either trivial or isomorphic to $\mathbb{Z}$.

Section 4. There are two natural topologies on $\operatorname{Im}\varphi$: the factor-topology of $C^\infty(M, \mathbb{R})$ and the induced topology of the ambient space $C^\infty(M, M)$. Assuming $\operatorname{Int}\operatorname{Fix}\Phi = \emptyset$, we prove that under mild conditions on fixed points of Φ, these topologies coincide (Theorem 17). In this case $\varphi: C^\infty(M, \mathbb{R}) \to \operatorname{Im}\varphi$ is either a homeomorphism or a covering map.

Section 5. We study the set $\Gamma_\Phi = \varphi^{-1}(\mathcal{D}(M))$. It is a union of two disjoint, open subsets Γ_Φ^+ and Γ_Φ^- corresponding to diffeomorphisms of M preserving and reversing (respectively) orientation of trajectories of Φ. We prove that Γ_Φ^+ and Γ_Φ^- are *convex* if regarded as subsets of the *linear* space $C^\infty(M,\mathbb{R})$ (Theorem 19). Also, if $f \in \operatorname{Im}\varphi$ and $z \in \operatorname{Fix}\Phi$, then the tangent map $f'(z)$ is an isomorphism whence $\operatorname{Im}\varphi \subset \mathcal{E}_0(\Phi)$.

Section 6. A sufficient condition for the relation $\operatorname{Im}\varphi = \mathcal{E}_0(\Phi)$ is given (Theorem 25). In this case we have $\varphi(\Gamma_\Phi^+) = \mathcal{D}_0(\Phi)$.

Sections 7 and 8. We show that Theorems 17 and 25 hold true for a flow Φ that satisfies the conditions of Theorem 1 (Theorem 27). This proves part (A) of Theorem 1.

Section 9. Assuming M is compact, we describe the homotopy type of the spaces $\mathcal{D}_0(\Phi)$ and $\mathcal{E}_0(\Phi)$. This completes Theorem 1.

Finally, in Section 10 we shortly discuss the closure of $\mathcal{E}_0(\Phi)$.

2. Preliminaries

2.1. Whitney topologies

In the sequel, all manifolds are assumed to be smooth (C^∞). Let us denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\overline{\mathbb{N}}_0 = \mathbb{N} \cup \{0, \infty\}$. For a subset X of a topological space Y the symbols $\overline{X}$, $\operatorname{Int} X$, and $\operatorname{Fr} X = \overline{X} \setminus \operatorname{Int} X$ mean the closure, the interior, and the frontier of X in Y, respectively.

Let M and N be manifolds, $f \in C^\infty(M,N)$, $x \in M$, $k \in \overline{\mathbb{N}}_0$. Denote by $[f]_x^k$ the k-jet of f at x. Let $J^k(M,N)$ be the manifold of k-jets from M to N and d be a metric on $J^k(M,N)$. For any $k \in \overline{\mathbb{N}}_0$, the space $C^\infty(M,N)$ can be endowed with the "weak" and "strong" Whitney topologies, which we denote by C^k_W and C^k_S, respectively (see, e.g., Mather [4], Hirsch [2]). Let us recall the definitions.

Let $f \in C^\infty(M,N)$. Then a base of the *weak* C^k_W-topology on $C^\infty(M,N)$ at f consists of sets of the form

$$\mathcal{N}_K^\varepsilon(f) = \bigl\{g \in C^\infty(M,N) \mid d\bigl([f]_x^k, [g]_x^k\bigr) < \varepsilon,\ \forall x \in K\bigr\}, \tag{1}$$

where $K \subset M$ is compact and $\varepsilon > 0$. For $k = 0$, this topology is often called *compact-open*. A base of the *strong* C^k_S-topology on $C^\infty(M,N)$ at f is generated by sets of the form

$$\mathcal{N}^\delta(f) = \bigl\{g \in C^0(A,B) \mid d\bigl([g]_x^k, [f]_x^k\bigr) < \delta(x),\ \forall x \in M\bigr\}, \tag{2}$$

where $\delta : M \to (0,\infty)$ is any continuous function.

Suppose A, B, A', and B' are manifolds, $\mathcal{X} \subset C^\infty(A,B)$, $F : \mathcal{X} \to C^\infty(A',B')$ is a map, $r, r' \in \overline{\mathbb{N}}_0$, and the symbols T and T' stand either for "W" or "S". We say that $\mathcal{X}$ is *C^r_T-open* (*-closed*, etc.) if it is open (closed) in the C^r_T-topology of $C^\infty(A,B)$. We say that F is *$C^{r,r'}_{T,T'}$-continuous* (*-homeomorphism*, *-embedding*) if F becomes continuous (a homeomorphism, an embedding) whenever $C^\infty(A,B)$ and $C^\infty(A',B')$ are endowed with the topologies C^r_T and $C^{r'}_{T'}$, respectively. If $r = r'$ and $T = T'$, then F is said to be *C^r_T-continuous*. Typical examples of C^r_T-continuous maps are given, e.g., in Mather [4].

2.2. Flows

Let $U \subset M$ be an open, connected set and $\mathcal{J} = (-a, a) \subset \mathbb{R}$, where $a > 0$. A *partial flow* on U is a smooth map

$$\Phi : U \times \mathcal{J} \to M$$

with the following properties. If $x \in U$ and $t, s \in \mathcal{J}$, then

(1) $\Phi(x, 0) = x$,
(2) $\Phi(\Phi(x,t), s) = \Phi(x, t+s)$ provided $\Phi(x,t) \in U$ and $t + s \in \mathcal{J}$.

If $U = M$ and $\mathcal{J} = \mathbb{R}$, then we say that Φ is *global*. By Φ_t denote the restriction of Φ to $U \times t$, where $t \in \mathcal{J}$. The *trajectory* of a point $x \in U$ is the set $\Phi(x \times \mathcal{J})$. There are three types of trajectories: *constant* (fixed point), *closed* or *periodic* (homeomorphic image of S^1), and *non-closed* (one-to-one image of $\mathbb{R}$). A point that is not fixed is called *regular*. The *period* of a periodic point x is the least positive number $\operatorname{Per} x$ such that $\Phi(x, \operatorname{Per} x) = x$.

The *Jordan cell* $J_p(A)$ of a $(k \times k)$-matrix A is the following $(pk \times pk)$-matrix:

$$J_p(A) = \left.\begin{Vmatrix} A & 0 & \cdots & 0 & 0 \\ E_k & A & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & E_k & A \end{Vmatrix}\right\} p,$$

where E_k is the unit $(k \times k)$-matrix. If $\alpha, \beta \in \mathbb{R}$, then we denote

$$R(\alpha, \beta) = \begin{Vmatrix} \alpha & -\beta \\ \beta & \alpha \end{Vmatrix}. \tag{3}$$

It is well known that each square matrix with real entries is a conjugate to a matrix of the form

$$\operatorname{diag}\bigl[J_{k_1}(\lambda_1), \ldots, J_{k_m}(\lambda_m), J_{p_1}\bigl(R(\alpha_1, \beta_1)\bigr), \ldots, J_{p_s}\bigl(R(\alpha_s, \beta_s)\bigr)\bigr], \tag{4}$$

where $\lambda_i \in \mathbb{R}$ $(i = 1, \ldots, m)$ and $\beta_j \neq 0$ $(j = 1, \ldots, s)$ (see, e.g., Theorem 2.2.5 in Palis and de Melo [6]). We also need the following formulas

$$e^{R(\alpha,\beta)t} = e^{\alpha t} \begin{Vmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{Vmatrix}, \tag{5}$$

$$e^{J_k(A)t} = \begin{Vmatrix} e^{At} & 0 & \cdots & 0 \\ t \cdot e^{At} & e^{At} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{t^{k-1}}{(k-1)!} \cdot e^{At} & \frac{t^{k-2}}{(k-2)!} \cdot e^{At} & \cdots & e^{At} \end{Vmatrix}. \tag{6}$$

3. Maps generated by smooth shifts

Let $\Phi : U \times \mathcal{J} \to M$ be a partial flow. Then Φ yields the *shift* mapping $\varphi : C^\infty(U, \mathcal{J}) \to C^\infty(U, M)$ defined by

$$\varphi(\alpha)(z) = \Phi\bigl(z, \alpha(z)\bigr), \tag{7}$$

where $\alpha \in C^\infty(U, \mathcal{J})$ and $z \in M$. If $\alpha \in C^\infty(U, \mathcal{J})$, then we say that $f = \varphi(\alpha)$ is a *shift along trajectories of Φ by α* and α is a *shift-function* of f.

Lemma 2. *For each $r \in \overline{\mathbb{N}}_0$ and $T =$ “W” or “S”, the map φ is C^r_T-continuous.*

Proof. Let $*: C^\infty(U, \mathcal{J}) \to \mathrm{id}_U \in C^\infty(U, M)$ be the constant map, where $\mathrm{id}_U: U \subset M$ is the identity embedding. Then φ coincides with the following composition:

$$C^\infty(U, \mathcal{J}) \xrightarrow{*\times \mathrm{id}} C^\infty(U, M) \times C^\infty(U, \mathcal{J})$$
$$\overset{\approx}{\to} \ C^\infty(U, M \times \mathcal{J}) \overset{\Phi_*}{\to} C^\infty(U, M),$$

where the first arrow is the product of $*$ and the identity map of $C^\infty(U, \mathcal{J})$, the second one is a natural C^r_T-homeomorphism, and the third one is induced by Φ and is C^r_T-continuous as well (e.g., Mather [4]). $\square$

Proposition 3. *If Φ is global, then the image* $\mathrm{Im}\,\varphi$ *is a semigroup and the intersection* $\mathrm{Im}\,\varphi \cap \mathcal{D}(M)$ *is a subgroup of* $C^\infty(M, M)$.

Proof. Suppose $\alpha, \beta, \gamma \in C^\infty(M, \mathbb{R})$, $f = \varphi(\alpha)$, $g = \varphi(\beta)$, $h = \varphi(\gamma)$, and $h \in \mathcal{D}(M)$. Let us show that $f \circ g$ and $h^{-1} \in \mathrm{Im}\,\varphi$. Consider the functions

$$\sigma_{f\circ g} = \beta + \alpha \circ g, \tag{8}$$
$$\sigma_{h^{-1}} = -\gamma \circ h^{-1}. \tag{9}$$

Then it can easily be seen that $\varphi(\sigma_{f\circ g}) = f \circ g$ and $\varphi(\sigma_{h^{-1}}) = h^{-1}$. $\square$

Definition 4. The set $Z_{\mathrm{id}}(\varphi) = \varphi^{-1}(\mathrm{id}_U)$, where $\mathrm{id}_U: U \subset M$ is the natural inclusion, will be called the *kernel* of φ.

Notice that, by formulas (8) and (9), φ is not a homomorphism. Nevertheless the following lemma explains our terminology.

Lemma 5. *Let $\alpha, \beta \in C^\infty(U, \mathcal{J})$ be functions such that $\alpha - \beta \in C^\infty(U, \mathcal{J})$. Then $\varphi(\alpha) = \varphi(\beta)$ if and only if $\alpha - \beta \in Z_{\mathrm{id}}$.*

Proof. The relation $\varphi(\alpha)(x) = \Phi(x, \alpha(x)) = \Phi(x, \beta(x)) = \varphi(\beta)(x)$ for $x \in U$ is equivalent to the following one: $\Phi(x, \alpha(x) - \beta(x)) = x$, i.e., $\alpha - \beta \in Z_{\mathrm{id}}$. $\square$

Corollary 6. *Suppose Φ is global. Then $Z_{\mathrm{id}} = \varphi^{-1}(\mathrm{id}_M)$ is a* subgroup *of $C^\infty(M, \mathbb{R})$ and $\varphi^{-1} \circ \varphi(\alpha) = \{\alpha + Z_{\mathrm{id}}\}$ for each $\alpha \in C^\infty(M, \mathbb{R})$. Thus there is a natural identification* $\mathrm{Im}\,\varphi \cong C^\infty(M, \mathbb{R})/Z_{\mathrm{id}}$.

3.1. Regular points

Let $y \in U$ be a regular point of Φ, $\alpha \in C^\infty(U, \mathcal{J})$, $f = \varphi(\alpha)$, and $a = \alpha(y)$. Then the point $z = f(y) = \Phi_a(y)$ is also regular. Hence there exists a neighborhood W of z and local coordinates $(x_1, \ldots, x_k)$ on W such that $z = 0$ and $\Phi(x_1, \ldots, x_k, t) = (x_1 + t, x_2, \ldots, x_k)$. Let $V = f^{-1}(W) \cap \Phi_{-a}(W)$ be a neighborhood of y. Then α can be expressed on V in terms of f as follows:

$$\alpha(x) = p_1 \circ f \circ \Phi(x, a) - p_1 \circ \Phi(x, a), \tag{10}$$

where $p_1 : \mathbb{R}^n \to \mathbb{R}$ is the projection onto the first coordinate. This relation will be often used. The first application is the local uniqueness of functions $\mu \in Z_{\mathrm{id}}$ at regular points of Φ. (Corollary 8.)

Lemma 7. *Let $\mu \in Z_{\mathrm{id}}$, and let ω be a non-constant trajectory of Φ. If ω is non-closed, then $\mu|_\omega = 0$. Otherwise, $\mu|_\omega = n\theta$, where $\theta = \operatorname{Per}\omega$ and $n \in \mathbb{Z}$.*

Proof. If ω is non-closed, then for any $x, y \in \omega$ there exists a *unique* number $t \in \mathcal{J}$ such that $y = \Phi(x,t)$. In particular, $t = 0$ iff $x = y$. Thus $\mu|_\omega \equiv 0$.

Let ω be closed, and $x \in \omega$. Then the relation $x = \Phi(x,t)$ holds iff $t = n\theta$ for some $n \in \mathbb{Z}$. Since μ is continuous and the set $\{n\theta\}$ is discrete, it follows that $\mu|_\omega$ is constant. □

Corollary 8. *Let C be a component of the set of regular points of Φ, $x \in C$ be a point, and $\alpha, \beta \in C^\infty(U, \mathcal{J})$ be such that $\varphi(\alpha) = \varphi(\beta)$. If $\alpha(x) = \beta(x)$, then $\alpha|_C = \beta|_C$. In particular, if $\alpha \in Z_{\mathrm{id}}$ and $\alpha(x) = 0$, then $\alpha|_C = 0$.*

Proof. See formula (10). □

3.2. Fixed points

Proposition 9. *Let $\alpha, \beta \in C^\infty(U, \mathcal{J})$ be such that $\alpha = \beta$ on $U \setminus \operatorname{Int}\operatorname{Fix}\Phi$. Then $\varphi(\alpha) = \varphi(\beta)$. In particular, $\alpha \in Z_{\mathrm{id}}$ iff $\beta \in Z_{\mathrm{id}}$.*

Proof. We must show that $\varphi(\alpha)(y) = \Phi(y, \alpha(y)) = \Phi(y, \beta(y)) = \varphi(\beta(y))$ for all $y \in U$. For $y \in U \setminus \operatorname{Fix}\Phi$ this holds by the condition $\alpha(y) = \beta(y)$. If $y \in \operatorname{Fix}\Phi$, then $\Phi(y,t) = y$ for all $t \in \mathbb{R}$. Hence $\Phi(y, \alpha(y)) = \Phi(y, \beta(y)) = y$. □

Proposition 10. *Let $z \in \operatorname{Fr}(\operatorname{Fix}\Phi)$ and $V = \bigcup_{\lambda \in \Lambda} V_\lambda$ be the union of components V_λ of $U \setminus \operatorname{Fix}\Phi$ such that $z \in \overline{V_\lambda}$ for all $\lambda \in \Lambda$. If $\mu \in Z_{\mathrm{id}}$, then each of the following conditions* (1) *and* (2) *implies that $\mu \equiv 0$ on $\overline{V}$.*

(1) *The tangent linear flow at z is trivial, i.e., $\frac{\partial\Phi}{\partial x}(z,t) = E_n$ for all $t \in \mathcal{J}$. In particular, this holds for any point $z \in \operatorname{Fr}(\operatorname{Int}\operatorname{Fix}\Phi)$.*
(2) $\mu(z) = 0$.

For the proof we need the following lemma. Let $M(n)$ be the space of real square $n \times n$ matrices, E_n be the unit $n \times n$ matrix, and $\exp : M(n) \to GL_n(\mathbb{R})$ be the exponential map.

Lemma 11. *Let $\{A_i\}_{i \in \mathbb{N}} \subset M(n)$ be a sequence of matrices such that for each $i \in \mathbb{N}$ the flow $\Phi_i(x,t) = e^{A_i t}x$ has at least one closed trajectory. Let θ_i be the minimum of periods of the closed trajectories of Φ_i. If $\lim_{i\to\infty} A_i = 0$, then $\lim_{i\to\infty}\theta_i = \infty$.*

Proof (*Sketch*). Let $A \in M(n)$ and $\Lambda = \{\lambda_k\}_{k=1}^r$ be the set of eigenvalues of A such that $\lambda_k = i\beta_k$ for $\beta_k \in \mathbb{R} \setminus \{0\}$. Then it is easily seen that the linear flow $\Phi(x,t) = e^{At}x$ has a closed trajectory iff $\Lambda \neq \emptyset$. In this case the period of any closed trajectory of Φ

is $\geqslant \min_{k=1,\dots,r} \frac{2\pi}{|\beta_k|}$. Now the lemma follows from the continuity of spectrums of linear operators. $\square$

Proof of Proposition 10. We will show that $\mu \equiv 0$ on each component V_λ. Suppose that V_λ contains a non-closed trajectory. Then $\mu|_{V_\lambda} \equiv 0$ by Corollary 8. Thus suppose that V_λ consists of periodic points only. Let $\{z_i\}_{i\in\mathbb{N}} \subset V_\lambda$ be a sequence of periodic points of Φ converging to z as $i \to \infty$. For each $i \in \mathbb{N}$ denote $\theta_i = \operatorname{Per} z_i$. By Lemma 7, $\mu(z_i) = n_i\theta_i$ for some $n_i \in \mathbb{Z}$. Then, by continuity of μ, we get $\mu(z_i) = n_i\theta_i \to \mu(z) < \infty$. Taking a subsequence of $\{\theta_i\}_{i\in\mathbb{N}}$ (if necessary) we can assume that there exists a finite or an infinite limit $\theta = \lim_{i\to\infty} \theta_i \geqslant 0$.

We prove that in both cases (1) and (2), $n_i = 0$ and therefore $\mu(z_i) = 0$ for all sufficiently large $i \in \mathbb{N}$. Since each point $z_i \in V_\lambda$ is regular, it follows that $\mu \equiv 0$ on V_λ. This will complete our proposition.

Define $\Psi : U \times \mathcal{J} \to GL_n(\mathbb{R})$ by $\Psi(x,t) = \frac{\partial\Phi}{\partial x}(x,t)$. Since $\Psi(x,0) = E_n$ for all $x \in U$, it follows that the map $\nu = \exp^{-1} \circ \Psi : U \times \mathcal{J} \to M(n)$ is well defined on some neighborhood of $(z,0)$ in $U \times \mathcal{J}$. Thus $\Psi(x,t) = e^{\nu(x,t)}$.

Notice that the restriction map $\Psi(x,*) : \mathcal{J} \to GL_n(\mathbb{R})$ is a local homomorphism, i.e., it yields a linear flow on $\mathbb{R}^n$. Hence $A(x,t) = A(x) = \nu(x,t)/t$ does not depend on $t \in \mathcal{J}$ and $\Psi(x,t) = e^{A(x)t}$.

We now show that for each periodic point x the flow $\Psi(x,*)$ has closed trajectories. Let $F(x) = \frac{\partial\Phi}{\partial t}(x,0)$ be the vector field generating Φ. Applying the operator $\frac{\partial}{\partial t}$ to both parts of the relation $\Phi(\Phi(x,t),s) = \Phi(\Phi(x,s),t)$ and then setting $s = 0$, we obtain

$$\frac{\partial\Phi}{\partial t}\bigl(\Phi(x,t),0\bigr) = \frac{\partial\Phi}{\partial x}(x,t)\frac{\partial\Phi}{\partial t}(x,0),$$

i.e., $F(\Phi(x,t)) = \Psi(x,t)F(x)$. Thus the vectors $F(\Phi(x,t))$ and $F(x)$ lie on same trajectory of the flow $\Psi(x,*)$. In particular, if x is a periodic point of Φ, then $\Psi(x,\operatorname{Per} x)F(x) = F(x)$, i.e., the vector $F(x)$ is a periodic point of $\Psi(x,*)$ and $\operatorname{Per} F(x) \leqslant \operatorname{Per}(x)$. Consider now the cases (1) and (2) of our proposition.

(1) Suppose $\frac{\partial\Phi}{\partial x}(z,t) = \Psi(z,t) = E_n$. Since $\Psi(z_i,t) \to \Psi(z,t) = E_n$, it follows from Lemma 11 that $\theta = \lim_{i\to\infty}\theta_i \geqslant \lim_{i\to\infty} \operatorname{Per} F(z_i) = \infty$. Since $\lim_{i\to\infty} n_i\theta_i < \infty$, we get $\lim_{i\to\infty} n_i = 0$.
(2) Let $\mu(z) = 0$. We claim that $\theta = \lim_{i\to\infty}\theta_i > 0$. Since $\lim_{i\to\infty} n_i\theta_i = \mu(z) = 0$, it will follow that $\lim_{i\to\infty} n_i = 0$. Thus suppose $\theta = 0$. Then

$$\frac{\partial\Phi}{\partial x}(z,t) \underset{i\to\infty}{\longleftarrow} \frac{\partial\Phi}{\partial x}(z_i,t) = \frac{\partial\Phi}{\partial x}\bigl(z_i, \theta_i\{t/\theta_i\}\bigr) \underset{i\to\infty}{\longrightarrow} \frac{\partial\Phi}{\partial x}(z_i,0) = E_n,$$

i.e., $\frac{\partial\Phi}{\partial x}(z,t) = E_n$ for all $t \in \mathcal{J}$, where $\{t\}$ is the fractional part of $t \in \mathbb{R}$. Then by (1) we get $\theta = \infty$, which contradicts the assumption $\theta = 0$. $\square$

Theorem 12.

(1) *Suppose* $\operatorname{Int}\operatorname{Fix}\Phi \neq \emptyset$. *Then*

$$Z_{\mathrm{id}} = \bigl\{\mu \in C^\infty(U,\mathcal{J}) \mid \mu|_{U\setminus \operatorname{Int}\operatorname{Fix}\Phi} = 0\bigr\}.$$

(2) *Let* $\operatorname{Int}\operatorname{Fix}\Phi = \emptyset$. *Then either* $Z_{\mathrm{id}} = \{0\}$ *or there exists a function* $\nu \in Z_{\mathrm{id}}$ *such that* $\nu(x) \neq 0$ *for all* $x \in U$ *and for any other* $\mu \in Z_{\mathrm{id}}$, *we have* $\mu = n\nu$, *where* $n \in \mathbb{Z}$. *Thus if* Φ *is global, then* Z_{id} *is either* 0 *or* $\mathbb{Z}$.

Proof. (1) Let $X = U \setminus \operatorname{Int}\operatorname{Fix}\Phi$ and $\mu \in C^\infty(U, \mathcal{J})$. If $\mu|_X = 0$, then by Proposition 9, we have $\mu \in Z_{\mathrm{id}}$.

Conversely, let $\mu \in Z_{\mathrm{id}}$. We will show that the set $Y = \mu^{-1}(0) \cap X$ is a nonempty, open-closed subset of X intersecting each component of X. This implies $\mu|_X = 0$ whence $Y = X$.

Evidently, Y is closed. Moreover, by (1) of Proposition 10, $\mu|_{\operatorname{Fr}(\operatorname{Int}\operatorname{Fix}\Phi)} = 0$. Thus $Y \supset \operatorname{Fr}(\operatorname{Int}\operatorname{Fix}\Phi) = \operatorname{Fr}(X) \neq \emptyset$. Since U is connected, it follows that $\operatorname{Fr}(\operatorname{Int}\operatorname{Fix}\Phi)$ (and therefore Y) intersects each component of X. Finally, let $z \in Y$. Then, from Corollary 8, we obtain that $\mu = 0$ in some neighborhood of z in X whenever z is regular. Suppose $z \in \operatorname{Fix}\Phi$. Then by (2) of Proposition 10, $\mu = 0$ in some neighborhood of z in X. Thus Y is open in X.

(2) Let $\operatorname{Int}\operatorname{Fix}\Phi = \emptyset$. For each $x \in U$ define $\tau_x : Z_{\mathrm{id}} \to \mathbb{R}$ by $\tau_x(\mu) = \mu(x)$, where $\mu \in Z_{\mathrm{id}}$. Then by Corollary 8 and by (2) of Proposition 10, τ_x is injective. Moreover, it follows from Lemma 7 that for each regular point x of Φ the set $\operatorname{Im}\tau_x$ is a discrete subset of $\mathbb{R}$ consisting of some integer multiples of some $\theta \in \mathbb{R}$. Suppose $Z_{\mathrm{id}} \neq \{0\}$. Since $\mathcal{J}$ is a connected neighborhood of $0 \in \mathbb{R}$, we see that there exists a number $g \in \operatorname{Im}\tau_x$ dividing all elements of $\operatorname{Im}\tau_x$. Then the function $\nu = \tau_x^{-1}(g)$ satisfies the statement of the theorem. □

Proposition 13. *Suppose* $\operatorname{Int}\operatorname{Fix}\Phi = \emptyset$. *Then each of the following two conditions implies that* $Z_{\mathrm{id}} = \{0\}$.

(1) *The tangent flow at some fixed point* $z \in \operatorname{Fix}\Phi$ *is trivial*;
(2) Φ *has at least one non-closed trajectory.*

Proof. Let $\mu \in Z_{\mathrm{id}}$. From (1) of Proposition 10 and Lemma 7 it follows that both conditions (1) and (2) imply that there exists a point $x \in U$ such that $\mu(x) = 0$. Since $\operatorname{Int}\operatorname{Fix}\Phi = \emptyset$, it follows from Theorem 12 that $\mu \equiv 0$ on U. □

4. Local sections of φ

Suppose that Φ is global. Then by Corollary 6, the map φ has the following decomposition:

$$\varphi : C^\infty(M, \mathbb{R}) \xrightarrow{\widetilde{\varphi}} C^\infty(M, \mathbb{R})/Z_{\mathrm{id}} \xrightarrow{j} \operatorname{Im}\varphi \subset C^\infty(M, M), \tag{11}$$

where $\widetilde{\varphi}$ is the factor-map and j is the bijection $\{\alpha + Z_{\mathrm{id}}\} \mapsto \varphi(\alpha)$.

Suppose $C^\infty(M, \mathbb{R})$ and $C^\infty(M, M)$ are endowed with some topologies. Recall that the corresponding factor-topology on $C^\infty(M, \mathbb{R})/Z_{\mathrm{id}}$ is defined as follows: a subset $W \subset C^\infty(M, \mathbb{R})/Z_{\mathrm{id}}$ is open iff $\widetilde{\varphi}^{-1}(W)$ is open in $C^\infty(M, \mathbb{R})$. Then j is continuous iff so is φ. In general, j is not a homeomorphism, i.e., the factor-topology of $\operatorname{Im}\varphi$ from $C^\infty(M, \mathbb{R})$ can differ from the induced topology of the ambient space $C^\infty(M, M)$. For the case $\operatorname{Int}\operatorname{Fix}\Phi = \emptyset$ we give a sufficient condition for j to be a homeomorphism in the related strong Whitney topologies. It requires existence of weakly continuous local sections of φ at each fixed point of Φ. For purposes of Theorem 1, we will consider flows depending on a parameter.

Let Φ be a partial flow and D^k be an open k-dimensional disk. Define the partial flow $\widetilde{\Phi} : (U \times D^k) \times \mathcal{J} \to M \times D^k$ as the product of Φ by the trivial flow on D^k, i.e., $\widetilde{\Phi}(x, \tau, t) = (\Phi(x, t), \tau)$, where $(x, \tau, t) \in U \times D^k \times \mathcal{J}$. For each open subset $V \subset U \times D^k$ formula (7) yields a *shift-map* $\varphi_V : C^\infty(V, \mathcal{J}) \to C^\infty(V, M)$ of $\widetilde{\Phi}$ by $\varphi_V(\alpha)(x, \tau) = \Phi(x, \tau, \alpha(x, \tau))$, where $(x, \tau) \in V$ and $\alpha \in C^\infty(V, M)$.

Proposition 14. *The map φ is* locally injective *in C^r_S-topology of $C^\infty(U, \mathcal{J})$ for any $r \in \overline{\mathbb{N}}_0$ iff* $\operatorname{Int}\operatorname{Fix}\Phi = \emptyset$. *In this case there exists a continuous function $\delta : U \to (0, \infty)$ such that for any open $V \subset U \times D^k$ and $\alpha \in C^\infty(V, \mathcal{J})$ the restriction of φ_V to the C^0_S-neighborhood*

$$\mathcal{M}^\delta_V = \left\{ \beta \in C^\infty(V, \mathcal{J}) \mid \left| \alpha(x, \tau) - \beta(x, \tau) \right| < \delta(x),\ \forall (x, \tau) \in V \right\} \tag{12}$$

of α is injective and $\mathcal{M}^\delta_V \cap \{\mathcal{M}^\delta_V + \mu\} = \emptyset$ for each $\mu \in Z_{\mathrm{id}}$ provided $\mu \neq 0$. Hence, if Φ is global, then $\widetilde{\varphi}$ is covering map.

Proof. Suppose $\operatorname{Int}\operatorname{Fix}\Phi \neq \emptyset$. Let $\alpha \in C^\infty(U, \mathcal{J})$ and let $\mathcal{N}$ be any C^r_S-neighborhood of α in $C^\infty(U, \mathcal{J})$. Then there exists $\beta \in \mathcal{N}$ such that $\alpha = \beta$ on $M \setminus \operatorname{Int}\operatorname{Fix}\Phi$, whence $\varphi(\alpha) = \varphi(\beta)$ by Lemma 9. Thus φ is not injective.

Let $\operatorname{Int}\operatorname{Fix}\Phi = \emptyset$. We will construct a function δ satisfying the statement of our proposition. From the proof of Proposition 10 it follows that for each $x \in U$ there exists a compact neighborhood W_x of x and a number $\tau_x > 0$ such that τ_x is less than the half of the period of any closed trajectory of Φ passing through W_x. If W_x intersects no periodic trajectories, then we set $\tau_x = 1$. Since M is paracompact, there exists a continuous function $\delta : U \to (0, 1)$ such that $\delta(x) < \tau_x$ for all $x \in U$. Then δ satisfies the statement of our proposition. □

Definition 15. Suppose $\operatorname{Int}\operatorname{Fix}\Phi = \emptyset$. A point $z \in U$ is said to be an $(\mathrm{S})^k$-point of Φ if for any sufficiently small open neighborhood $V \subset U \times D^k$ of $(z, 0)$ with compact closure $\overline{V}$, and any function $\alpha \in C^\infty(V, \mathcal{J})$ there exists a C^0_W-neighborhood $\mathcal{M} \subset C^\infty(V, \mathcal{J})$ of α such that the restriction of φ_V to $\mathcal{M}$ is injective and the inverse map $(\varphi_V)^{-1} : \varphi_V(\mathcal{M}) \to \mathcal{M}$ is C^r_W-continuous for each $r \in \mathbb{N}_0$. A point is (S) if it is $(\mathrm{S})^k$ for each $k \in \mathbb{N}_0$.

Let us explain this definition in more details. From Proposition 14, we see that for any open neighborhood $V \subset U \times D^k$ of $(z, 0)$ there exists a C^0_W-neighborhood $\mathcal{M} \subset C^\infty(V, \mathcal{J})$ of α such that the restriction of φ_V to $\mathcal{M}$ is injective (e.g., we may put $\mathcal{M} = \mathcal{M}^\delta_V$). Thus, on $\mathcal{M}$, the relation $g(x, \tau) = \Phi(x, \tau, \beta(x, \tau))$ is equivalent to $\beta(x, \tau) = \Psi(x, \tau, g)$, where Ψ is some function. Then a point z is $(\mathrm{S})^k$ whenever Ψ induces a continuous map $\varphi_V(\mathcal{M}) \to \mathcal{M}$ in the *weak* C^r-topologies. In particular, this holds whenever Ψ is smooth in (x, τ) and continuously depend on all partial derivatives of g near $(z, 0)$ up to order r. For instance, the following lemma is a direct corollary of formula (10).

Lemma 16. *Any regular point of a flow is* (S).

Theorem 17. *Suppose* $\operatorname{Int}\operatorname{Fix}\Phi = \emptyset$ *and each fixed point* $z \in \operatorname{Fix}\Phi$ *of* Φ *is* $(\mathrm{S})^0$. *For any* $\alpha \in C^\infty(U, \mathcal{J})$ *let*

$$\mathcal{M}^\delta = \mathcal{M}^\delta(\alpha) = \left\{\beta \in C^\infty(U, \mathcal{J}) \mid \left|\beta(x) - \alpha(x)\right| < \delta(x),\ \forall x \in U\right\}$$

be a C^0_S*-neighborhood of* α *in* $C^\infty(U, \mathcal{J})$ *such that the restriction* $\varphi|_{\mathcal{M}^\delta}$ *is injective. Then the inverse map* $\varphi^{-1} : \varphi(\mathcal{M}^\delta) \to \mathcal{M}^\delta$ *is* C^r_S*-continuous for any* $r \in \overline{\mathbb{N}}_0$. *Hence, for global* Φ, *the map* $\varphi : C^\infty(M, \mathbb{R}) \to \operatorname{Im}\varphi$ *is a covering map in the* C^r_S*-topologies for any* $r \in \overline{\mathbb{N}}_0$.

For the proof we need the following statement.

Lemma 18. *Let* M *and* N *be smooth manifolds,* $f \in C^\infty(M, N)$, *and* $r \in \mathbb{N}_0$. *Let also* $\{U_i\}_{i\in\Lambda}$ *be a locally finite family of open subsets of* M *and for each* $i \in \Lambda$ *let* $\mathcal{U}_i$ *be a* C^r_W*-neighborhood of the restriction* $f|_{U_i}$ *in the space* $C^\infty(U_i, N)$. *Define* $p_i : C^\infty(M, N) \to C^\infty(U_i, N)$ *by* $f \mapsto f|_{U_i}$ *for* $f \in C^\infty(M, N)$, *and let* $\widetilde{\mathcal{U}}_i = p_i^{-1}(\mathcal{U}_i)$. *Then* $\widetilde{\mathcal{U}} = \bigcap_{i\in\Lambda}\widetilde{\mathcal{U}}_i$ *is a* C^r_S*-neighborhood of* f *in* $C^\infty(M, N)$.

Proof. Since p_i is C^r_W-continuous, it follows that for each $i \in \Lambda$ the set $\widetilde{\mathcal{U}}_i$ contains a base C^r_W-neighborhood $\mathcal{N}^{\varepsilon_i}_{K_i}(f|_{U_i})$ of $f|_{U_i}$ defined by formula (1), where $K_i \subset U_i$ is compact and $\varepsilon_i > 0$. Note that $\{K_i\}_{i\in\Lambda}$ is locally finite and M is paracompact. So there exists a continuous function $\delta : M \to (0, \infty)$ such that $\delta(x) < \varepsilon_i$ for $x \in K_i$. Let $\widetilde{\mathcal{N}} = \mathcal{N}^\delta(f)$ be the open C^r_S-base neighborhood of f in $C^r(M, N)$ defined by formula (2). Then, obviously, $\widetilde{\mathcal{N}} \subset \widetilde{\mathcal{U}}$. □

Proof of Theorem 17. From Lemma 16 and the assumption of the theorem we obtain that each $z \in U$ is an $(\mathrm{S})^0$-point of Φ. Let $r \in \mathbb{N}_0$, d be a metric on $J^r(U, \mathcal{J})$, and $\hat{\delta} : U \to (0, \infty)$ be a continuous function such that the base C^r_S-neighborhood $\mathcal{M}^{\hat{\delta}} = \{\beta \in C^\infty(U, \mathcal{J}) \mid d([\alpha]^r_x, [\beta]^r_x) < \hat{\delta}(x), \forall x \in U\}$ of α (see formula (2)) lies in $\mathcal{M}^\delta$. Hence $\varphi|_{\mathcal{M}^{\hat{\delta}}}$ is injective. Let us prove that $\varphi(\mathcal{M}^{\hat{\delta}})$ contains a C^r_S-neighborhood of $f = \varphi(\alpha)$ in $\operatorname{Im}\varphi$, i.e., there exists an open neighborhood $\widetilde{\mathcal{N}}$ of f in $C^\infty(U, M)$ such that $\operatorname{Im}\varphi \cap \widetilde{\mathcal{N}} \subset \varphi(\mathcal{M}^{\hat{\delta}})$. Since $\hat{\delta}$ can be chosen arbitrary small, we will get that φ^{-1} is C^r_S-continuous on $\varphi(\mathcal{M}^\delta)$.

Since U is paracompact, there exist two at most countable locally finite coverings $\{U_i\}_{i\in\Lambda}$ and $\{K_i\}_{i\in\Lambda}$ of U such that U_i is open and $K_i \subset U_i$ is compact. Moreover, using the $(\mathrm{S})^0$ property of points, we may assume that for each $i \in \Lambda$ there exists a C^0_W-neighborhood $\mathcal{M}_i$ of $\alpha|_{U_i}$ in $C^\infty(U_i, \mathcal{J})$ such that the restriction of φ_{U_i} to $\mathcal{M}_i$ is a C^r_W-embedding for all $r \in \mathbb{N}_0$.

Let d_i be a metric on $J^r(U, \mathcal{J})$ and $\varepsilon_i = \inf_{x\in K_i}\hat{\delta}(x)$. Define a C^r_W-neighborhood $\mathcal{N}^{\varepsilon_i}_{K_i}(\alpha|_{U_i})$ of $\alpha|_{U_i}$ by formula (1), with $d = d_i$, and set $\mathcal{M}'_i = \mathcal{N}^{\varepsilon_i}_{K_i}(\alpha|_{U_i}) \cap \mathcal{M}_i$. Then the image $\mathcal{N}_i = \varphi_{U_i}(\mathcal{M}'_i)$ is a C^r_W-neighborhood of $f|_{U_i}$ in $C^\infty(U_i, M)$ for all $r \in \mathbb{N}_0$.

Let $p_i : C^\infty(U, M) \to C^\infty(U_i, M)$ be the "restriction to U_i" mapping and $\widetilde{\mathcal{N}}_i = p_i^{-1}(\mathcal{N}_i)$. Since $\{U_i\}_{i\in\Lambda}$ is locally finite, it follows from Lemma 18 that the intersection $\bigcap_{i\in\Lambda}\widetilde{\mathcal{N}}_i$ contains some C^r_S-open neighborhood $\widetilde{\mathcal{N}}$ of f. One can easily verify that

$\widetilde{\mathcal{N}} \cap \operatorname{Im}\varphi \subset \varphi(\mathcal{M}^{\hat{\delta}})$. This proves the theorem. For the convenience of the reader we give the following commutative diagram illustrating our constructions.

$$\begin{array}{ccccccc}
\mathcal{M}^{\hat{\delta}} & \subset & C^\infty(U,\mathcal{J}) & \longrightarrow & C^\infty(U_i,\mathcal{J}) & \supset & \mathcal{M}_i' = \mathcal{N}_{K_i}^{\varepsilon_i}(\alpha|_{U_i}) \cap \mathcal{M}_i \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_{U_i}} & & \downarrow{\scriptstyle \varphi_{U_i}} \\
\varphi(\mathcal{M}^{\hat{\delta}}) & \subset & C^\infty(U,M) & \xrightarrow{p_i} & C^\infty(U_i,M) & \supset & \varphi_{U_i}(\mathcal{M}_i') = \mathcal{N}_i.
\end{array}$$

5. Shifts that are diffeomorphisms

The following Theorem 19 gives a precise description of the set

$$\Gamma_\Phi = \varphi^{-1}\big(\mathcal{D}(M)\big) = \varphi^{-1}\big(\operatorname{Im}\varphi \cap \mathcal{D}(M)\big) \subset C^\infty(M,\mathbb{R}).$$

Theorem 19. *Suppose that Φ is a global flow on M, $F(z) = \frac{\partial \Phi}{\partial t}(z,0)$ be the vector field generating Φ and $\alpha \in C^\infty(M,\mathbb{R})$. Then $f = \varphi(\alpha)$ is a diffeomorphism of M if and only if f is proper ($f^{-1}(K)$ is compact for each compact $K \subset M$) and the following inequality holds at each $z \in M$:*

$$d\alpha\big(F(z)\big) \neq -1. \tag{13}$$

Moreover, f preserves orientation of trajectories iff $d\alpha(F(z)) > -1$.

Proof. The necessity is implied by the following lemma. □

Lemma 20. *The tangent map $f'(z) : TM_z \to TM_z$ is an isomorphism if and only if (13) holds true at z.*

Proof. We can assume $\alpha(z) = 0$. Otherwise, set $\beta = \alpha - \alpha(z)$ and $g = \varphi(\beta)$. Then $\beta(z) = 0$, $d\beta(F) = d\alpha(F)$, and $g'(z)$ is an isomorphism iff so is $f'(z)$.

So let $\alpha(z) = 0$. Then $f(z) = z$. Let us choose a local chart at z such that $z = 0 \in \mathbb{R}^n$ and calculate the determinant $|f'(z)|$ of $f'(z)$. We claim that

$$\big|f'(z)\big| = 1 + d\alpha\big(F(z)\big). \tag{14}$$

This will prove our lemma. For simplicity we consider only the case $n = 2$. The general case is analogous. Let $\Phi = (A, B)$, where $A(x, y; t)$ and $B(x, y; t)$ are coordinate functions of Φ at this chart. Then

$$\begin{aligned}
\big|f'(z)\big| &= \big|\Phi'\big(z;\alpha(z)\big)\big| = \begin{vmatrix} A_x + A_t\alpha_x & A_y + A_t\alpha_y \\ B_x + B_t\alpha_x & B_y + B_t\alpha_y \end{vmatrix} \\
&= \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} + \begin{vmatrix} A_t & A_y \\ B_t & B_y \end{vmatrix} \alpha_x + \begin{vmatrix} A_x & A_t \\ B_x & B_t \end{vmatrix} \alpha_y \\
&= \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} (1 + X\alpha_x + Y\alpha_y),
\end{aligned}$$

where, by Cramer's formulas, the vector (X, Y) is a solution of the following linear equation

$$\begin{Vmatrix} A_x & A_y \\ B_x & B_y \end{Vmatrix} \begin{Vmatrix} X \\ Y \end{Vmatrix} = \begin{Vmatrix} A_t \\ B_t \end{Vmatrix}.$$

Since $\alpha(z) = 0$, we have

$$\Phi_{\alpha(z)} = \begin{Vmatrix} A_x & A_y \\ B_x & B_y \end{Vmatrix} = \mathrm{id} \quad \text{and} \quad (A_t, B_t) = F(z),$$

whence $(X, Y) = F(z)$. It remains to note that $X\alpha_x + Y\alpha_y = d\alpha(X, Y) = d\alpha(F(z))$. □

Sufficiency. Suppose f is proper and (13) holds at each $z \in M$. Then f' is non-degenerate at each $z \in M$ and it remains to show that f is bijective. Since $f(\omega) \subset \omega$ for each trajectory ω of Φ, we should establish that $f(\omega) = \omega$ and $f|_\omega$ is one-to-one. This is evident for fixed points. Let ω be a regular trajectory. Since f' is non-degenerate, it follows that the restriction $f|_\omega : \omega \to \omega$ is a proper map having no critical points and homotopic to a diffeomorphism. Hence $f|_\omega$ is one-to-one. □

Corollary 21. *Let $z \in \operatorname{Fix}\Phi$ and $\alpha \in C^\infty(M, \mathbb{R})$. Then the map $\varphi(\alpha)$ is a local diffeomorphism near z.*

Proof. Since $F(z) = 0$, we see that $d\alpha(F(z)) = 0 > -1$. □

5.1. Decomposition of Γ_Φ

Define the subsets Γ_Φ^+ and Γ_Φ^- of $C^\infty(M, \mathbb{R})$ by

$$\Gamma_\Phi^+ = \bigl\{\alpha \in \Gamma_\Phi \mid d\alpha\bigl(F(x)\bigr) > -1,\ \forall x \in M\bigr\},$$
$$\Gamma_\Phi^- = \bigl\{\alpha \in \Gamma_\Phi \mid d\alpha\bigl(F(x)\bigr) < -1,\ \forall x \in M\bigr\}.$$

Then $\Gamma_\Phi^+ \cap \Gamma_\Phi^- = \emptyset$. Moreover, from Theorem 19 we get $\Gamma_\Phi = \Gamma_\Phi^+ \cup \Gamma_\Phi^-$.

Lemma 22. *For each $r \in \overline{\mathbb{N}}_0$ the sets Γ_Φ^+ and Γ_Φ^- are C_S^r-open in $C^\infty(M)$ and* convex *if regarded as subsets of the* linear space *$C^\infty(M)$.*

Proof. Since $\mathcal{D}(M)$ is C_S^r-open in $C^\infty(M, M)$, and φ and differentiating along the vector field are C_S^r-continuous, we obtain that Γ_Φ^+ and Γ_Φ^- are also open. Let us prove that Γ_Φ^+ is convex. The proof for Γ_Φ^- is analogous.

Let $\alpha_0, \alpha_1 \in \Gamma_\Phi^+$, $\alpha_s = s\alpha_0 + (1-s)\alpha_1$, and $f_s = \varphi(\alpha)$ for $s \in [0, 1]$. Then $d\alpha_s(F) = sd\alpha_0 + (1-s)d\alpha_1 > -1$, whence $f_s'(x)$ is an isomorphism for each $x \in M$. By the arguments similar to the proof of sufficiency in Theorem 19, f_s is injective. Let us show that f_s is onto.

First consider the flow $\Omega : (M \times \mathbb{R}) \times \mathbb{R} \to M \times \mathbb{R}$ on $M \times \mathbb{R}$ defined by the formula $\Omega(x, t, s) = (x, t + s)$. Then the *mapping $\Phi : M \times \mathbb{R} \to M$* gives rise to the factorization of the flow Ω onto the *flow Φ*, i.e., $\Phi \circ \Omega_s = \Phi_s \circ \Phi$ for all $s \in \mathbb{R}$. Indeed,

$$\Phi \circ \Omega_s(x, t) = \Phi(x, t + s) = \Phi\bigl(\Phi(x, t), s\bigr) = \Phi_s \circ \Phi(x, t).$$

Let $\alpha \in C^\infty(M,\mathbb{R})$, $f(x) = \Phi(x,\alpha(x))$, $\tilde\alpha = \alpha \circ \Phi : M\times\mathbb{R} \to \mathbb{R}$, and $\tilde f : M\times\mathbb{R} \to M\times\mathbb{R}$ be the shift along trajectories of Ω by the function $\tilde\alpha$, i.e., $\tilde f(x,t) = \Omega(x,t,\tilde\alpha(x,t)) = (x, t+\alpha\circ\Phi(x,t))$. Then it is easily seen that

$$\Phi \circ \tilde f = f \circ \Phi. \tag{15}$$

Now let us define $\tilde\alpha_s = \alpha_s \circ \Phi$ and $\tilde f_s(x,t) = \Omega(x,t,\tilde\alpha_s(x,t))$. Then

$$\tilde f_s(x,t) = \bigl(x, t+\tilde\alpha_s(x,t)\bigr) = \bigl(x, s\bigl(t+\tilde\alpha_0(x,t)\bigr) + (1-s)\bigl(t+\tilde\alpha_1(x,t)\bigr)\bigr). \tag{16}$$

By assumption, f_0 and f_1 are onto. Then, by (15), so are $\tilde f_0$ and $\tilde f_1$. It follows from formula (16) that so is $\tilde f_s$ for each $s \in [0,1]$. Hence, by (15), f_t is onto. □

Let us illustrate Lemmas 20 and 22 by applying them to the flow $\Phi(x,t) = x+t$ on $\mathbb{R}$. Let $\alpha \in C^\infty(\mathbb{R},\mathbb{R})$ and $f(x) = \Phi(x,\alpha(x)) = x+\alpha(x)$. Then the Lie derivative of α with respect to Φ coincides with the usual derivative α'. Hence the inequality $\alpha'(z) \neq -1$ is equivalent to $f'(z) \neq 0$ (compare with Lemma 20). Notice that $\mathbb{R}$ is a unique trajectory of Φ. So the space of diffeomorphisms of $\mathbb{R}$ preserving (reversing) the orientation of this trajectory coincides with the space of smooth monotone functions with positive (negative) derivative. Therefore, these spaces are open and convex (compare with Lemma 22).

6. Describing the image Im φ

Let Φ be a *global* flow on M. Define the subset $\mathcal{E}(\Phi) \subset C^\infty(M,M)$ to be consisting of maps f such that

(1) $f(\omega) \subset \omega$ for each trajectory ω of Φ,
(2) $f'(z) : TM_z \to TM_z$ is an isomorphism for each $z \in \operatorname{Fix}\Phi$.

By definition, put $\mathcal{D}(\Phi) = \mathcal{E}(\Phi) \cap \mathcal{D}(M)$ and let $\mathcal{E}_0(\Phi)$ and $\mathcal{D}_0(\Phi)$ be the identity path components of the corresponding spaces $\mathcal{E}(\Phi)$ and $\mathcal{D}(\Phi)$ in the C^0_W-topologies. The next lemma follows from Corollary 21 and definitions.

Lemma 23. $\operatorname{Im}\varphi \subset \mathcal{E}_0(\Phi)$ *and* $\varphi(\Gamma^+_\Phi) \subset \mathcal{D}_0(\Phi)$.

Definition 24. Let D^k be the k-dimensional disk, $k \in \mathbb{N}_0$. A point $z \in \operatorname{Fix}\Phi$ is an (E)^k-point of Φ if there exists an open neighborhood V of z such that the following holds true:

Suppose $\alpha : (V \setminus \operatorname{Fix}\Phi) \times D^k \to \mathbb{R}$ is a C^∞-function such that the mapping $f : (V \setminus \operatorname{Fix}\Phi) \times D^k \to M$ defined by $f(x,t) = \Phi(x,\alpha(x,t))$ has a C^∞-extension onto $V \times D^k$. Then α has a C^∞-extension onto $V \times D^k$.

A point z is (E) if it is (E)^k for any $k \in \mathbb{N}_0$.

Theorem 25. *Suppose* $\operatorname{Int}\operatorname{Fix}\Phi = \emptyset$ *and each fixed point of* Φ *is* (E)^0. *Then*

$$\operatorname{Im}\varphi = \mathcal{E}_0(\Phi), \tag{17}$$

$$\varphi\bigl(\Gamma^+_\Phi\bigr) = \mathcal{D}_0(\Phi). \tag{18}$$

Proof. It is easy to see that (17) implies (18). So let us prove (17). Note that the C^0_W-topologies of the spaces $C^\infty(M,\mathbb{R})$ and $C^\infty(M,M)$ coincide with the compact-open ones. This allows us to consider homotopies instead of continuous maps from I into these spaces.

Let $f \in \mathcal{E}_0(\Phi)$. We will show that there exists a function $\alpha \in C^\infty(M,\mathbb{R})$ such that $f = \varphi(\alpha)$. By definition of $\mathcal{E}_0(\Phi)$, there exists a continuous map $\nu : I \to C^\infty(M,M)$ such that $\nu(0) = \mathrm{id}_M$ and $\nu(1) = f$. We will construct a map $\tilde\nu : I \to C^\infty(M,\mathbb{R})$ such that $\nu = \varphi \circ \tilde\nu$. Then $f = \nu(1) = \varphi \circ \tilde\nu(1) \in \operatorname{Im}\varphi$. Since $\nu(0) = \mathrm{id}_M$, we can set $\tilde\nu(0) = 0$.

In the "homotopy" language, the previous paragraph means that there exists a homotopy $F : M \times I \to M$ such that

$$F(x,t) = \nu(t)(x), \quad F(x,0) = x \quad \text{and} \quad F(x,1) = f(x).$$

Our aim is to construct a homotopy $\tilde\Lambda : M \times I \to \mathbb{R}$ such that $F(x,t) = \varphi(\tilde\nu(t))(x) = \Phi(x, \tilde\Lambda(x,t))$ and $\tilde\Lambda(x,0) = 0$. We have that $\tilde\Lambda$ is defined on $M \times 0$ by $\tilde\Lambda(x,t) = x$ and we must extend it to $M \times I$.

Let ω be a regular trajectory of Φ, $x \in \omega$, and $p : \mathbb{R} \to \omega$ be defined by $p(t) = \Phi(x,t)$. Note that $F(x \times I) \subset \omega$ and $F(x,0) = x$. Then there exists a *unique* function $\tilde\Lambda_x : I \to \mathbb{R}$ such that

$$\tilde\Lambda_x(0) = 0 \quad \text{and} \quad F(x,t) = p \circ \tilde\Lambda_x(t) = \Phi\bigl(x, \tilde\Lambda_x(t)\bigr) \quad \text{for all } t \in I.$$

Indeed, suppose ω is homeomorphic to the circle S^1. Then p is a covering map and our statements follow from the covering homotopy property for p. If ω is a non-closed trajectory, then p is continuous and bijective, though possibly not a homeomorphism. Nevertheless the restriction of p to any compact subset of $\mathbb{R}$ is a homeomorphism, as being a continuous and injective map from a compact space into a Hausdorff space. So we put $\tilde\Lambda_x = p^{-1} \circ F$.

Define $\tilde\Lambda$ on $(M \setminus \operatorname{Fix}\Phi) \times I$ by $\tilde\Lambda(x,t) = \tilde\Lambda_x(t)$. Then by formula (10), where α can depend on a parameter, $\tilde\Lambda(x,t)$ is C^∞ in x for each $t \in I$.

Thus for each $t \in I$ the C^∞-map F_t has the C^∞ shift-function $\tilde\Lambda_t$ defined on $M \setminus \operatorname{Fix}\Phi$. Let $x \in \operatorname{Fix}\Phi$. Then by the condition $(\mathrm{E})^0$ for x, $\tilde\Lambda_t$ can be C^∞-extended onto some neighborhood of x remaining a shift-function for F_t. Since $\operatorname{Fix}\Phi$ is nowhere dense in M, all these extensions are coherent and yield a well-defined C^∞-extension of F_t. In particular $f = F_1 = \varphi(\Lambda_1)$. □

6.1. *A point that is not* $(\mathrm{E})^0$

Consider the differential equation on $\mathbb{R}$: $\frac{dx}{dt} = x^n$ $(n \geqslant 2)$ and let Φ be the corresponding local flow defined on the interval $\mathcal{I} = (-a,a)$, $a > 0$. Evidently Φ has exactly three trajectories: $(-a,0)$, 0 and $(0,a)$. We will show that the origin 0 is not a $(\mathrm{E})^0$-point of Φ.

Proof. Note that the space $\mathcal{E}(\mathcal{I},\Phi)$ consists of C^∞-functions f on $\mathcal{I}$ preserving the sign of points and such that $f'(0) > 0$. Therefore it is path connected in C^0_W-topology, i.e., $\mathcal{E}_0(\mathcal{I},\Phi) = \mathcal{E}(\mathcal{I},\Phi)$. Let $f \in \mathcal{E}(\mathcal{I},\Phi)$. Then $f(0) = 0$ and $f'(0) > 0$. Therefore, by the Hadamard lemma (see formula (27)), $f(z) = zg(z)$, where g is a unique C^∞-function on $\mathcal{I}$ such that $g(0) = f'(0) > 0$.

Let us calculate the time $\alpha(z)$ between points z and $f(z)$, where $z \in \mathcal{I}$.

$$\alpha(z) = \int_z^{f(z)} \mathrm{d}t = \int_z^{f(z)} \frac{\mathrm{d}x}{x^n} = \frac{z^{n-1} - f(z)^{n-1}}{(n-1) f(z)^{n-1} z^{n-1}}$$
$$= \frac{1 - g(z)^{n-1}}{(n-1) f(z)^{n-1}} = \frac{1-g}{z^{n-1}} \cdot \frac{1 + g + g^2 + \cdots + g^{n-2}}{(n-1) g^{n-1}}.$$

It follows that α is C^∞ at 0 if, and only if, $f = z + z^n h(z)$, where h is a C^∞-function on $\mathbb{R}$ (equivalently $f(0) = 0$, $f'(0) = 1$ and $f^{(k)}(0) = 0$ for $k = 2, \ldots, n-1$). Thus for each $n \geqslant 2$, we have $\operatorname{Im}\varphi \neq \mathcal{E}_0(\mathcal{I}, \Phi)$. □

Note that in these cases the flow Φ is not linear. We will prove in the next section that $\operatorname{Im}\varphi = \mathcal{E}_0(\mathcal{I}, \Phi)$ for linear flows.

7. Regular factors and extensions of flows

We prove here that fixed points of “regular” extensions of linear flows are (S) and (E). Let us represent $\mathbb{R}^{m+n}$ as $\mathbb{R}^m \times \mathbb{R}^n$ and denote its points by (x, y), where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. Let also U^m and U^n be open disks in $\mathbb{R}^m$ and $\mathbb{R}^n$ (respectively) with centers at the origins and $U^{m+n} = U^m \times U^n$.

Definition 26. Let $\Phi : U^{m+n} \times \mathcal{J} \to \mathbb{R}^{m+n}$ and $\Psi : U^m \times \mathcal{J} \to \mathbb{R}^m$ be partial flows and $p_m : \mathbb{R}^{m+n} \to \mathbb{R}^m$ be the natural projection. Then Ψ is a *regular factor* of Φ and Φ is, in turn, a *regular extension* of Ψ whenever for each $t \in \mathcal{J}$ the following condition holds:

$$p_m \circ \Phi_t = \Psi_t \circ p_m. \tag{19}$$

Flows Φ and Ψ are *regularly equivalent* when they are regular factors of each other. A flow Φ is *regularly minimal* if it is nonconstant and each of its regular nonconstant factors is regularly equivalent to Φ.

Rewriting Φ in the coordinates (x, y, t) of $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$ we see that (19) is equivalent to the following representation:

$$\Phi(x, y, t) = \bigl(\Psi(x, t), B(x, y, t)\bigr), \tag{20}$$

where $B : U^{m+n} \times \mathcal{J} \to \mathbb{R}^n$ is a C^∞-map. Thus Ψ is the “former” coordinate function of Φ and does not depend on y.

It follows from the Hadamard lemma that any C^∞-map $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(0) = 0$ can be represented in the form $f(x) = A(x) \cdot x$, where $x \in \mathbb{R}^n$, and $A(x)$ is a C^∞ $(m \times n)$-matrix. Suppose now that in Definition 26 the origin 0 is a fixed point of Φ. Then formula (20) can also be rewritten in the matrix form as

$$\Phi(x, y, t) = \begin{Vmatrix} P(x,t) & 0 \\ Q(x,y,t) & R(x,y,t) \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \end{Vmatrix}, \tag{21}$$

where P, Q, R are C^∞-matrices of dimensions $m \times m$, $m \times n$ and $n \times n$, respectively, such that P does not depend on y. Hence $\Psi(x, t) = P(x, t)x$.

Theorem 27. *Let $\Phi : U^{m+n} \times \mathcal{J} \to \mathbb{R}^{m+n}$ be a nontrivial partial flow such that the origin $z_{m+n} = 0 \in \mathbb{R}^{m+n}$ is a fixed point of Φ. Suppose there exists a linear flow $\Psi(x,t) = e^{At}x$ on $\mathbb{R}^m$ such that Φ is a regular extension of Ψ at z_{m+n}. Then z_{m+n} is* (S) *and is an* (E)*-point of Φ.*

Proof. First we show that the properties (S) and (E) are inherited by regular extensions (Lemmas 28 and 29). Then we prove them for linear flows. So let Φ and Ψ be the flows of Definition 26, $z_m = 0 \in \mathbb{R}^m$ and $z_{m+n} = 0 \in \mathbb{R}^{m+n}$ be the origins, D^k be an open k-disk, $U^{n+k} = U^n \times D^k$ and $U^{m+n+k} = U^m \times U^n \times D^k$. □

Lemma 28. *The origin z_{m+n} is an* (S)*-point for Φ whenever so is z_m for Ψ.*

Proof. Let $V \subset U^{m+n+k}$ be an open neighborhood of $(z_{m+n}, 0)$ with compact closure $\overline{V}$ and $\alpha \in C^\infty(V,\mathbb{R})$. Then the following diagram is commutative:

$$\begin{tikzcd} C^\infty(V,\mathbb{R}) \arrow[r, equal] \arrow[d, "\phi"'] & C^\infty(V,\mathbb{R}) \arrow[d, "\psi"] \\ C^\infty(V,\mathbb{R}^{m+n}) \arrow[r, "P_m"] & C^\infty(V,\mathbb{R}^m) \end{tikzcd}$$

where $\phi(\alpha)(x,y,s) = \Phi(x,y,\alpha(x,y,s))$, $\psi(\alpha)(x,y,s) = \Psi(x,\alpha(x,y,s))$, and $P_m(h) = p_m \circ h$ for $\alpha \in C^\infty(V,\mathbb{R})$ and $h \in C^\infty(V,\mathbb{R}^{m+n})$.

Let $\delta_\phi : U^{m+n} \to (0,\infty)$ and $\delta_\psi : U^m \to (0,\infty)$ be functions satisfying the statement of Proposition 14 for the flows Φ and Ψ respectively and such that $\delta_\phi(x,y) \leqslant \delta_\psi(x)$. Define two C^0_S-neighborhoods of α in the space $C^\infty(V,\mathbb{R})$ by

$$\mathcal{M}_\psi = \bigl\{\beta \in C^\infty(V,\mathbb{R}) \mid \bigl|\alpha(x,y,s) - \beta(x,y,s)\bigr| < \delta_\psi(x)\bigr\},$$
$$\mathcal{M}_\phi = \bigl\{\beta \in C^\infty(V,\mathbb{R}) \mid \bigl|\alpha(x,y,s) - \beta(x,y,s)\bigr| < \delta_\phi(x,y)\bigr\}.$$

Then $\mathcal{M}_\phi \subseteq \mathcal{M}_\psi$ and the restrictions $\psi|_{\mathcal{M}_\psi}$ and $\phi|_{\mathcal{M}_\phi}$ are injective. Hence the inverse mapping $\phi^{-1} : \phi(\mathcal{M}_\phi) \to \mathcal{M}_\phi$ coincides with the composition $\psi^{-1} \circ P_m$. By the condition $(\mathrm{S})^{n+k}$ for z_m with respect to Ψ, the inverse map $\psi^{-1} : \psi(\mathcal{M}_\psi) \to \mathcal{M}_\psi$ is C^r_W-continuous for all $r \in \mathbb{N}_0$. Since P_m is also C^r_W-continuous, we see that so is $\phi^{-1} = \psi^{-1} \circ P_m$. □

Lemma 29. *The origin z_{m+n} is an* (E)*-point for Φ whenever so is z_m for Ψ.*

Proof. Let $\alpha \in C^\infty(U^{m+n+k}, \mathcal{J})$ be such that the map

$$f(x,y,s) = \Phi\bigl(x,y,\alpha(x,y,s)\bigr) = \bigl(\Psi\bigl(x,\alpha(x,y,s)\bigr), B(x,y,s)\bigr)$$

has a C^∞-extension to U^{m+n+k}. We will show that α has a C^∞-extension to U^{m+n+k}. First note that

$$\bigl(U^m \setminus \operatorname{Fix}\Psi\bigr) \times \bigl(U^{n+k}\bigr) \subset \bigl(U^{m+n} \setminus \operatorname{Fix}\Phi\bigr) \times U^k. \tag{22}$$

Indeed, it is obvious that $\operatorname{Fix}\Phi \subset \operatorname{Fix}\Psi \times U^n$. Then

$$\bigl(U^m \setminus \operatorname{Fix}\Psi\bigr) \times U^n \ \subset\ U^{m+n} \setminus \operatorname{Fix}\Phi.$$

Multiplying both sides of this relation by U^k we get (22). Since z_m is an $(\mathrm{E})^{n+k}$-point of Ψ, we obtain that α has a C^∞-extension onto U^{m+n+k}. $\square$

To complete the theorem it remains to prove that for each nontrivial linear flow Ψ on $\mathbb{R}^m$ the origin z_m is (E) and (S). Notice that we may consider regularly minimal linear flows only. They are described by the following lemma. The proof is immediate and will be omitted.

Lemma 30. *A nonconstant linear flow $\Psi(x,t) = e^{At}x$ is regularly minimal iff the matrix A is a conjugate to one of the following matrices*:

(1) $J_1(\lambda) = \|\lambda\|,\ \lambda \neq 0,\ \Psi(x,t) = xe^{\lambda t},\ x \in \mathbb{R};$
(2) $R(\alpha,\beta) = \left\| \begin{smallmatrix} \alpha & -\beta \\ \beta & \alpha \end{smallmatrix} \right\|,\ \beta \neq 0,\ \Psi(z,t) = ze^{(\alpha+\beta i)t},\ z \in \mathbb{C};$
(3) $J_2(0) = \left\| \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right\|,\ \Psi(x,y,t) = (x+ty, y),\ (x,y) \in \mathbb{R}^2.$

Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$, U be an open neighborhood of the origin $0 \in \mathbb{F}$, $\mathring{U} = U \setminus \{0\}$ be a "punctured" neighborhood of 0, Ψ be a linear flow on $\mathbb{F}$ generated by one of the corresponding matrices (1)–(3) of Lemma 30 (in the case (3) we identify $\mathbb{C}$ with $\mathbb{R}^2$), and ψ be the corresponding shift-map of Ψ.

Let $\sigma : \mathring{U} \times D^k \to \mathbb{R}$ be a C^∞-function such that

$$h(x,\tau) = \psi(\sigma)(x,\tau) = \Psi\big(x,\tau,\sigma(x,\tau)\big)$$

is C^∞ on $U \times D^k$. We will show that σ can be C^∞-extended to $U \times D^k$ and obtain explicit formulas expressing σ in terms of h. They will imply the properties (S) and (E). The proof is based on two lemmas.

Lemma 31. *Let Ψ be a flow as in cases* (1) *or* (2) *of Lemma* 30. *Then there exists a unique smooth function $\gamma : U \times D^k \to \mathbb{F}$ such that $h(z,\tau) = z \cdot \gamma(z,\tau)$ and $\gamma(0,\tau) \neq 0$ for all $\tau \in D^k$.*

Lemma 32. *The map $Z : C^\infty(U \times D^k, \mathbb{F}) \to C^\infty(U \times D^k, \mathbb{F})$ defined by $Z(h)(z,\tau) = z \cdot h(z,\tau)$ is a C^r_W-embedding* (*i.e., a homeomorphism onto the image in the C^r_W-topologies*) *for each $r \in \mathbb{N}_0$.*

Assuming that these lemmas hold consider the following cases of Ψ.

(1) We have $h(x,\tau) = \Psi(x,\sigma(x,\tau)) = xe^{\lambda\sigma(x,\tau)}$ for $x \neq 0$. Since $0 \in \operatorname{Fix}\Psi$, it follows that $h(0,\tau) = 0$. From Lemma 31 we get $h(x,\tau) = x\gamma(x,\tau)$. Hence $\gamma(x,\tau) = e^{\lambda\sigma(x,\tau)}$ and $\gamma(0,\tau) = h'_x(0,\tau) > 0$. Thus

$$\sigma(x,\tau) = \frac{1}{\lambda}\ln\gamma(x,\tau). \tag{23}$$

(2) Denote $\omega = \alpha + i\beta$. Then $h(z,\tau) = ze^{\omega\sigma(z,\tau)}$ for $z \neq 0$. Using Lemma 31 we get $h(z,\tau) = z\gamma(z,\tau)$, where γ is C^∞. Hence $\gamma(z,\tau) = e^{\omega\sigma(z,\tau)}$. Now we separate the cases of α. If $\alpha \neq 0$, then

$$\sigma(z,\tau) = \frac{1}{2\alpha}\ln\bigl(\gamma(z,\tau)\overline{\gamma(z,\tau)}\bigr) = \frac{1}{2\alpha}\ln\bigl|\gamma(z,\tau)\bigr|^2. \tag{24}$$

Suppose $\alpha = 0$. Then $\gamma(z,\tau) = e^{i\beta\sigma(z,\tau)}$. It follows that σ is not unique and is determined up to a constant summand,

$$\sigma(z,\tau) = \frac{1}{\beta}\arg\bigl(\gamma(z,\tau)\bigr) + \frac{2\pi k}{\beta}, \quad k \in \mathbb{Z}. \tag{25}$$

(3) In this case, $\mathrm{Fix}\,\Psi = \{(x,0) \mid x \in \mathbb{R}\}$ and $h(x,y,\tau) = \Psi(x,y,\sigma(x,y,\tau)) = (x + y\sigma(x,y,\tau), y)$ for $y \neq 0$. Let $h_1(x,y,t) = x + y\sigma(x,y,\tau)$ be the first coordinate function of h. Then $\sigma = (h_1 - x)/y$ for $y \neq 0$. Note that the function $H(x,y,\tau) = h_1(x,y,\tau) - x$ is C^∞ and $H(x,0,\tau) \equiv 0$. Therefore we can apply the Hadamard lemma to H and obtain a unique C^∞ function $\gamma : U \times D^k \to \mathbb{R}$ such that $H(x,y,\tau) = y\gamma(x,y,\tau)$. Hence

$$\sigma(x,y,\tau) = \frac{h_2(x,y,\tau) - x}{y} = \gamma(x,y,\tau). \tag{26}$$

From formulas (23)–(26) we see that in all cases the function σ has a C^∞-extension to some neighborhood of $0 \in \mathbb{F}$. This implies the condition (E) for Ψ at $0 \in \mathbb{F}$. Furthermore, let V be any small open neighborhood V of 0. It follows from Lemma 32 and these formulas that there exists a C^0_W-neighborhood of $h|_V$ in $C^\infty(V,\mathbb{R})$ such that the correspondence $h|_V \mapsto \gamma|_V \mapsto \sigma|_V$ is C^r_W-continuous. Hence 0 is an (S)-point of Ψ. To complete the proof Theorem 27, we must prove Lemmas 31 and 32.

Proof of Lemma 32. Note that Z is linear, injective, and C^r_W-continuous for any $r \in \mathbb{N}_0$. Let us verify the C^r_W-continuity of the inverse map Z^{-1}.

For each compact set $K \subset U \times D^k$ and $r \in \mathbb{N}_0$, consider the norm $\|\cdot\|_{r,K}$ on $C^\infty(U \times D^k, \mathbb{F})$ defined by $\|h\|_{r,K} = \sum_{i=0}^r \sup_{x\in K} |D^i h(x)|$, where $|D^i h(x)|$ denotes the sum of absolute values of all derivatives of h of degree i. If K runs over all compact subsets of $U \times D^k$, the norms $\|\cdot\|_{r,K}$ generate the C^r_W-topology in $C^\infty(U \times D^k, \mathbb{F})$. Let $h \in C^\infty(U \times D^k, \mathbb{F})$ and $\gamma = Z(h) = zh$. The following inequality implies our lemma and can be easily verified:

$$\|\gamma\|_{r,K} \leqslant |z|\|h\|_{r-1,K} + \|h\|_{r,K} \leqslant \bigl(1 + |z|\bigr)\|h\|_{r,K} \leqslant (1 + \operatorname{diam} K)\|h\|_{r,K}. \qquad \square$$

8. Proof of Lemma 31

(1) In this case the lemma follows from the well-known Hadamard lemma. Indeed, since $h(x,\tau) = e^{\lambda\sigma(x,\tau)}x$ is C^∞, we see that $h(0,\tau) = 0$ for all τ. Then

$$h(x,\tau) = \int_0^x \frac{\partial h}{\partial t}(t,\tau)\,\mathrm{d}t = x\int_0^1 \frac{\partial h}{\partial t}(t \cdot x, \tau)\,\mathrm{d}t. \tag{27}$$

Denoting the last integral by $\gamma(x,\tau)$, we get that $h(x,\tau) = x\gamma(x,\tau)$, γ is C^∞, and $\gamma(0,\tau) = h'(0,\tau) \neq 0$ for each $\tau \in D^k$.

(2) Let us denote $\omega = \alpha + i\beta$. Then $h(z,\tau) = e^{\omega\sigma(z,\tau)}z$, so we must put

$$\gamma(z,\tau) = e^{\omega\sigma(z,\tau)}, \quad \forall z \neq 0. \tag{28}$$

Hence

$$\sigma = \frac{1}{2\alpha}\ln|\gamma|^2 = \frac{1}{\beta}\arg\gamma, \quad \forall z \neq 0. \tag{29}$$

Lemma 33. *The function γ satisfies the following equation*:

$$\mathrm{Im}(\omega\gamma\,\mathrm{d}\bar{\gamma}) = 0. \tag{30}$$

Proof. From (28) we get $\mathrm{d}\gamma = \omega\gamma\,\mathrm{d}\sigma$. Multiplying both sides of this formula by $\mathrm{d}\bar{\gamma}$ and taking into account that $\mathrm{d}\sigma$ and $\mathrm{d}\gamma\,\mathrm{d}\bar{\gamma}$ are real, we see that so is $\omega\gamma\,\mathrm{d}\bar{\gamma}$. □

To complete the proof of our lemma we separate the cases $\alpha \neq 0$ and $\alpha = 0$.

Lemma 34. *If $\alpha \neq 0$, then the functions σ and γ are C^∞.*

Proof. It suffices to prove that $|\gamma(z)|^2$ is C^∞. Indeed, since h is a diffeomorphism at 0, there exist constants c and C such that $0 < c < |h(z)|/|z| = |\gamma(z)| < C$ in some neighborhood of $0 \in \mathbb{C}$. Thus, if $|\gamma|^2$ is C^∞, then by (29) and (28) so are σ and γ.

Now, let us expand formula (30),

$$\omega\gamma\,\mathrm{d}\bar{\gamma} = \omega\frac{h}{z}\,\mathrm{d}\left(\frac{\bar{h}}{\bar{z}}\right) = \frac{\omega h}{z}\cdot\frac{\bar{z}\,\mathrm{d}\bar{h} - \bar{h}\,\mathrm{d}\bar{z}}{\bar{z}^2} = \frac{z\bar{z}\cdot\omega h\,\mathrm{d}\bar{h} - h\bar{h}\cdot\omega z\,\mathrm{d}\bar{z}}{(z\bar{z})^2}.$$

Since $z\bar{z}$, $h\bar{h}$ are real, the relation (30) is equivalent to the following one:

$$h\bar{h}\cdot\mathrm{Im}(\omega z\,\mathrm{d}\bar{z}) = z\bar{z}\cdot\mathrm{Im}(\omega h\,\mathrm{d}\bar{h}),$$

whence

$$|\gamma|^2\cdot\mathrm{Im}(\omega z\,\mathrm{d}\bar{z}) = \mathrm{Im}(\omega h\,\mathrm{d}\bar{h}). \tag{31}$$

Substituting $\mathrm{d}\bar{z} = \bar{\omega} = \alpha - i\beta$ in the last formula we get $\mathrm{Im}(\omega z\bar{\omega}) = y|\omega|^2$. Then the left side of Eq. (31) becomes equal to $|\gamma(x,y)|^2\cdot y|\omega|^2$. This function is C^∞ and so is the right-hand side. It follows from the Hadamard lemma that $|\gamma|^2$ is also C^∞. □

Suppose now that $\alpha = 0$. Then $\Psi(z,t) = e^{i\beta t}z$ and $\beta \neq 0$. Therefore,

$$z\bar{z} = h\bar{h}. \tag{32}$$

In fact, this is just another expression of (30) for our case $\alpha = 0$.

Claim 35. $\frac{\partial^n h}{\partial\bar{z}^n}(0) = 0$ *for each* $n = 1, 2, \ldots$.

For the proof we need the following lemma.

Lemma 36. *Let $h:\mathbb{R}^n \to \mathbb{R}^n$ be a C^1-diffeomorphism such that $h(0)=0$. Let $h'(0):\mathbb{R}^n \to \mathbb{R}^n$ be the tangent map of h at 0, and $g:\mathbb{R}^n \to \mathbb{R}$ be a continuous homogeneous function of degree k, i.e., $g(tx)=t^k g(x)$ for $t>0$ and $x \in \mathbb{R}^n$. If $g=g\circ h$ then $g=g\circ h'(0)$.*

Proof. Let $x \in \mathbb{R}^n$ and $t>0$. Then

$$g(x)=\frac{g(tx)}{t^k}=\frac{g(h(tx))}{t^k}=g\left(\frac{h(tx)}{t}\right)\underset{t\to 0}{\longrightarrow} g\circ h'(0)(x). \qquad \square$$

Proof of Claim 35. Let $h(z)=p(z)+iq(z)$, where $p,q\in C^\infty(\mathbb{C},\mathbb{R})$. We will use the induction in n. Let $n=1$. Then the relation (32) means that h preserves the homogeneous polynomial $\beta(x,y)=x^2+y^2$. It follows from Lemma 36 that so does $h'(0)$. Hence $h'(0)$ is an orthogonal matrix and $\frac{\partial h}{\partial \bar z}(0)=0$. Thus $h'_z(0)$ coincides with multiplication by e^{ia}, where $a\in[0,2\pi)$, whence $h(z)=e^{ia}z+\varepsilon_2$, where $\varepsilon_2=\mathrm{o}(|z|^2)$. Let us define $\gamma(0)=e^{ia}$. Then γ becomes continuous at 0.

Suppose we have proved the lemma for $n-1$. Then

$$h(z)=e^{ia}z+Az+B\bar z^n+\varepsilon_{n+1},$$

where A is a polynomial in the variables z and $\bar z$, $1\leqslant \deg A\leqslant n-1$, and $B=\frac{\partial^n h}{\partial \bar z^n}(0)$. Substituting h in formula (32) we obtain

$$z\bar z=h\bar h=z\bar z+Az\bar z+B\bar z^{n+1}+\bar A z\bar z+\overline{B}z^{n+1}+\theta_{n+2},$$

where $\theta_{n+2}=\mathrm{o}(|z|^{n+2})$. Hence $Az\bar z+B\bar z^{n+1}+\bar A z\bar z+\overline{B}z^{n+1}=-\theta_{n+2}$. Denote the left-hand side of this equality by η. Then η is a polynomial of degree $\leqslant n+1$ in the variables z and $\bar z$. On the other hand, $\eta=-\theta_{n+2}=\mathrm{o}(|z|^{n+2})$. Then

$$\eta=\overline{B}z^{n+1}+\big(A+\bar A\big)z\bar z+B\bar z^{n+1}\equiv 0.$$

Thus all coefficients of η are zeros. Since the first two summands contain a multiple of z, we obtain that the coefficient at $\bar z^n$ is B. Hence $B=0$. $\square$

The following lemma is left to the reader.

Lemma 37. *Let $\varepsilon:\mathbb{C}\to\mathbb{C}$ be a C^∞-function such that $\varepsilon=\mathrm{o}(|z|^k)$. Define $\tau:\mathbb{C}\to\mathbb{C}$ by $\tau(z)=\varepsilon(z)/z$ for $z\neq 0$ and $\tau(0)=0$. Then τ is C^{k-2}.*

Now we can complete the case (2). It follows from Claim 35 that for each $n\in\mathbb{N}$ the Taylor expansion of degree n of h at 0 has the form $h(z)=\nu_{n-1}z+\varepsilon_{n+1}$, where ν_{n-1} is a polynomial of degree $n-1$ in the variables z and $\bar z$ and $\varepsilon_{n+1}=\mathrm{o}(|z|^{n+1})$. Hence $\gamma(z)=h(z)/z=\nu_{n-1}+\varepsilon_{n+1}/z$. Applying Lemma 37 to the remainder ε_{n+1}/z, we see that this function is C^{n-1}. Hence so is γ for any $n\in\mathbb{N}$, i.e., γ is C^∞. Lemma 31 is proved. $\square$

9. Proof of Theorem 1

Let Φ be a global flow on M such that for each fixed point z of Φ there exist local coordinates $(x_1,\dots,x_n)$ and a nontrivial linear flow Ψ on $\mathbb{R}^m$ $(0<m\leqslant n)$ such that

$z = 0$ and for all t in some neighborhood of $0 \in \mathbb{R}$ we have $p_m \circ \Phi_t = \Psi_t \circ p_m$, where $p_m : \mathbb{R}^n \to \mathbb{R}^m$ is a natural projection.

Then $\operatorname{Int}\operatorname{Fix}\Phi = \emptyset$, since this holds for linear flows. Moreover, it follows from Theorem 27 that Φ satisfies the conditions (E) and (S) at each point $z \in \operatorname{Fix}\Phi$. Then by Theorems 17 and 25 we obtain that $\operatorname{Im}\varphi = \mathcal{E}_0(\Phi)$, the map $\varphi : C^\infty(M, \mathbb{R}) \to \operatorname{Im}\varphi$ is covering, and its group of covering slices is either 0 or $\mathbb{Z}$. Finally, by Theorem 25 and Lemma 22, the set $\Gamma^+_\Phi = \varphi^{-1}(\mathcal{D}_0(\Phi))$ is a convex subset of the linear space $C^\infty(M, \mathbb{R})$.

Suppose now that M is compact. Let X denote either $C^\infty(M, \mathbb{R})$ or Γ^+_Φ. Then the image $Y = \varphi(X)$ is either $\mathcal{E}_0(\Phi)$ or $\mathcal{D}_0(\Phi)$, respectively. Evidently X is a Fréchet manifold, whence so is Y, as being its image under the covering map φ. It follows that Y has the homotopy type of a CW-complex (e.g., Palais [5]). Since X is contractible, we get that Y is aspherical, i.e., $\pi_n(Y) = 0$ for all $n \geqslant 2$. By Theorem 12, $\pi_1(Y)$ is either 0 or $\mathbb{Z}$. Thus Y is either contractible or homotopically equivalent to S^1.

Since $Z_{\mathrm{id}} = \varphi^{-1}(\mathrm{id}_M) \subset \Gamma^+_\Phi$, we see that the embedding $\mathcal{D}_0(\Phi) \subset \mathcal{E}_0(\Phi)$ induces an isomorphism of all homotopy groups and is, therefore, a homotopy equivalence. Finally, suppose that either Φ has at least one non-closed trajectory, or the tangent linear flow at some fixed point of Φ is trivial. Then by Proposition 13, $Z_{\mathrm{id}} = 0$, whence $\mathcal{D}_0(\Phi)$ and $\mathcal{E}_0(\Phi)$ are contractible.

10. The closure of $\mathcal{E}_0(\Phi)$

By Theorem 1, in most cases the sets $\mathcal{E}_0(\Phi)$ and $\mathcal{D}_0(\Phi)$ are contractible. Nevertheless, as the referee of this paper noted, their closures are likely not so. For example, in the article [3] of J. Keesling the homeomorphisms group G of a solenoid Σ is considered. The path-component C of the unit element e in Σ is a dense one-parametric subgroup $\phi : \mathbb{R} \to \Sigma$ such that ϕ is one-to-one. Then C and $\overline{C} = \Sigma$ are of different homotopy types. It is proven that the identity path component G_{id} of G is homotopically equivalent to C while the closure of G_{id} has the homotopy type of Σ.

Notice that for each closed subset $K \subset M$ the set

$$\operatorname{Inv}(K) = \left\{ f \in C^\infty(M, M) \mid f(K) \subset K \right\}$$

is closed in $C^\infty(M, M)$ with the topology of point-wise convergence. Therefore it is closed in each Whitney topology of $C^\infty(M, M)$. Thus, if Φ is a global flow on M, then the set $\operatorname{Inv}(\Phi) = \bigcap_\omega \operatorname{Inv}(\overline{\omega})$, where ω runs over all trajectories of Φ, is closed in the Whitney topologies of $C^\infty(M, M)$. Clearly, $\mathcal{E}(\Phi) \subset \operatorname{Inv}(\Phi)$. Hence $\mathcal{E}_0(\Phi) \subset \operatorname{Inv}_0(\Phi)$, where $\operatorname{Inv}_0(\Phi)$ is the identity path component of $\operatorname{Inv}(\Phi)$ in compact-open topology. The following lemma is not hard to prove.

Lemma 38. *Let Φ be a* global *flow defined on a* connected *subset of* $\mathbb{R}$. *Then* $\overline{\mathcal{E}_0(\Phi)} = \operatorname{Inv}_0(\Phi)$.

However, in general, it seems to be a problem to prove this equality as well as to establish that $\operatorname{Inv}_0(\Phi)$ is closed. Consider, for instance, an irrational flow Φ on the n-torus T^n. Each trajectory of Φ is everywhere dense in T^n, whence $\operatorname{Inv}(\Phi) = C^\infty(T^n, T^n)$. Let d

be a metric on $C^\infty(T^n, T^n)$ yielding the compact-open topology. Since T^n is ANR, it follows that two continuous mappings $f, g : T^n \to T^n$ are homotopic provided $d(f, g)$ is sufficiently small. Therefore each path-component of $C^\infty(T^n, T^n)$ is open. Hence it is also closed as the complement to the union of all other ones. Thus $\mathrm{Inv}_0(\Phi)$ is closed.

Notice that Φ has no fixed points. Therefore it satisfies the conditions of Theorem 1, whence $\mathrm{Im}\,\varphi = \mathcal{E}_0(\Phi)$. Thus the statement $\overline{\mathcal{E}_0(\Phi)} = \mathrm{Inv}_0(\Phi)$ would mean that $\overline{\mathrm{Im}\,\varphi}$ is the unity path-component of $C^\infty(T^n, T^n)$, i.e., that for any $\varepsilon > 0$ each smooth mapping $f : T^n \to T^n$ that is homotopic to id_{T^n} could be ε-approximated in metric d by a map of the form $f_\varepsilon(x) = \Phi(x, \alpha_\varepsilon(x))$, where $\alpha_\varepsilon \in C^\infty(T^n, \mathbb{R})$. The author does not know whether this is true or not. One of the difficulties is exposed by the following general proposition: if $f(z)$ does not belong to the trajectory of z, then roughly speaking, $\lim_{\varepsilon\to 0} \alpha_\varepsilon(z) = \infty$.

Proposition 39. *Let Φ be a global flow on M. Suppose that there exists a trajectory ω of Φ such that $\mathrm{Int}\,\overline{\omega} \neq \emptyset$. Let also $f \in \overline{\mathrm{Im}\,\varphi} \setminus \mathrm{Im}\,\varphi$, and $z \in \mathrm{Int}\,\overline{\omega}$ be such that $f(z)$ does not belong to the trajectory ω_z of z. If $\{t_i\}_{i\in\mathbb{N}}$ is a sequence of reals such that $\lim_{i\to\infty} \Phi(z, t_i) = f(z)$, then $\lim_{i\to\infty} t_i = \infty$.*

Proof. First note that ω_z is non-closed. Denote $y = f(z)$ and $y_i = \Phi(z, t_i)$ for all $i \in \mathbb{N}$. Fix any $A > 0$ and define the compact subset $\omega_A \subset \omega_z$ by $\omega_A = \Phi(z \times [-A, A])$. Then $y \notin \omega_A$. Hence there exists a neighborhood U of y such that $U \cap \omega_A = \emptyset$. Since $\lim_{i\to\infty} y_i = y$ and $y_i \in \omega$, we have $y_i = \Phi(z, t_i) \in \omega_z \setminus \omega_A$ whence $t_i > A$ for almost all i. Taking the number A arbitrary large, we obtain that $\lim_{i\to\infty} t_i = \infty$. $\square$

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